# Math 3280 Tutorial 10

Recall:

① $X, Y$ have a joint probility mass function $p(x,y)$, then $E[g(X,Y)] = \sum_y \sum_x g(x,y) \cdot p(x,y)$

$X, Y$ have a joint prob density function $f(x,y)$, then

$$E[g(X,Y)] = \iint g(x,y) \cdot f(x,y)\, dx\, dy.$$

② $Cov(X,Y) = E[(X-E(X)) \cdot (Y-E[Y])]$

$= E[XY] - E[X] \cdot E[Y]$.

If $X, Y$ are independent r.v.s, $\Rightarrow Cov(X,Y) = 0$.

Example 1. If the joint probability density function of $X, Y$ is

$$f(x,y) = \frac{1}{L^2}, \quad 0 < x < L, \; 0 < y < L.$$

Compute $E[|X-Y|]$.

Solution:

$$E[|X-Y|] = \int_0^L \int_0^L |x-y| \cdot \frac{1}{L^2}\, dx\, dy$$

$$= \int_0^L \int_y^L (x-y) \frac{1}{L^2}\, dx\, dy + \int_0^L \int_0^y \frac{y-x}{L^2}\, dx\, dy$$

$$= \int_0^L \frac{(y-L)^2}{2L^2}\, dy + \int_0^L \frac{y^2}{2L^2}\, dy$$

$$= \frac{L^3}{6L^2} + \frac{L^3}{6L^2}$$

$$= \frac{L}{3}$$

Example 2. Compute the variance of a binomial r.v. $X$ with parameters $n$ and $p$.

Solution:

$$X = X_1 + X_2 + \cdots + X_n$$

where $X_i$ are independent Bernoulli r.v.s s.t.

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th trial is a success} \\ 0 & \text{o.w.} \end{cases}$$

Since $X_1, X_2 \cdots X_n$ are independent,

$$Var(X) = \sum_{i=1}^{n} Var(X_i).$$

$$Var(X_i) = E[X_i^2] - (E[X_i])^2 = E[X_i] - (E[X_i])^2 = p - p^2$$

$$Var(X) = \sum_{i=1}^{n} p(1-p)$$

$$= np(1-p).$$

3. Let $I_A$, $I_B$ be the indicator variables for the events $A$ and $B$. That is

$I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{o.w.} \end{cases}$ , $I_B = \begin{cases} 1 & \text{if } B \text{ occurs} \\ 0 & \text{o.w.} \end{cases}$

Prove that

$$Cov(I_A, I_B) = P(B)\cdot(P(A|B) - P(A)).$$

Solution:

$$Cov(I_A, I_B) = E[I_A \cdot I_B] - E[I_A]\cdot E[I_B]$$

$$E[I_A] = 1\cdot P(A) + 0\cdot(1-P(A))$$
$$= P(A).$$

$$E[I_B] = P(B).$$

$A \cap B$

$I_A \cdot I_B = \begin{cases} 1, & \text{if } AB \text{ occurs} \\ 0, & 0. \end{cases}$

$$E[I_A \cdot I_B] = 1\cdot P(AB) + 0\cdot(1-P(AB))$$
$$= P(AB)$$

$$Cov(I_A, I_B) = P(AB) - P(A)\cdot P(B) \qquad (P(AB) = P(A|B)\cdot P(B))$$
$$= P(A|B)\cdot P(B) - P(A)\cdot P(B)$$
$$= P(B)\cdot(P(A|B) - P(A)).$$

4. If $E[X] = 1$, $Var(X) = 5$. find

(a) $E[(X+2)^2]$.

(b) $Var(4+3X)$.

(c). If $X$, $Y$ are independent and indentically distributed, find $E[(X-Y)^2]$.

Solution:

(a) $E[(X+2)^2] = E[X^2 + 4X + 4]$

$$E[X^2] = Var(X) + (E[X])^2$$
$$= 5 + 1^2$$
$$= 6$$

$$= E[X^2] + 4 \cdot E[X] + 4$$
$$= 6 + 4 \times 1 + 4$$
$$= 14.$$

(b) $Var(4+3X)$
$$= E[(4+3X)^2] - (E[4+3X])^2.$$
$$= E[9 \cdot X^2 + 24X + 16] - (4 + 3 \cdot E(X))^2$$
$$= 9 \times 6 + 24 \times 1 + 16 - 4^2 - 24 \cdot E[X] - (E[X])^2$$
$$= 45.$$

(Alternatively, $Var(X+c) = Var(X)$ if $c$ is a constant)
$$Var(4+3X) = Var(3X)$$
$$= 9 \cdot Var(X)$$
$$= 45.$$

(c) $E[Y] = E[X] = 1$, $Var(Y) = Var(X) = 5$.
$$E[(X-Y)^2] = E[X^2 - 2XY + Y^2]$$
$$= E[X^2] - 2 \cdot E[XY] + E[Y^2]$$
$E[X^2] = 6$
$$= 6 - 2 \cdot E[X] \cdot E[Y] + 6$$
$E[Y^2] = 6$
$$= 6 - 2 \times 1 \times 1 + 6$$
$E[XY] = E[X] \cdot E[Y]$
$$= 10.$$
since $X, Y$ independent

5. If $X_1, X_2 \cdots X_n$ are independent and indentically distributed r.v.s having uniform distribution on $[0,1]$.

find $E[\max(X_1, X_2, \cdots, X_n)]$.

Solution:

Denote $X = \max(X_1, X_2, \cdots, X_n)$. then $X \in (0,1)$.

For $x \in (0,1)$.

$$P(X \le x) = P(\max(X_1, X_2, \cdots, X_n) \le x)$$
$$= P(X_1 \le x, X_2 \le x, \cdots, X_n \le x)$$
$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdots P(X_n \le x)$$
$$= x \cdot x \cdots x$$
$$= x^n$$

density of $X$: $\frac{dP(X \le x)}{dx} = n \cdot x^{n-1} \quad (x \in (0,1))$

$$E[X] = \int_0^1 x n x^{n-1} dx = n \cdot \int_0^1 x^n dx = \frac{n}{n+1}.$$

Alternatively. Since $X$ is non-negative, $X \in (0,1)$.

$$E[X] = \int_0^1 P(X > x) dx$$
$$= \int_0^1 (1 - x^n) dx$$
$$= 1 - \int_0^1 x^n dx$$
$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$